

Accuracy of minimal and optimal qubit tomography for finite-length experiments

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Practical quantum state tomography is usually performed by carrying out repeated measurements on many copies of a given state. The accuracy of the reconstruction depends strongly on the dimensionality of the system and the number of copies used for the measurements. We investigate the accuracy of an experimental implementation of a minimal and optimal tomography scheme for one- and two-qubit states encoded in the polarization of photons. A suitable statistical model for the attainable accuracy is introduced.

I. INTRODUCTION

The accurate characterization of quantum states and their evolution is central to quantum information processing. Quantum tomography or state estimation attempts to extract as much information about the state as possible out of physically realizable measurements on an ensemble of identically prepared copies. Qubits are the simplest carriers of quantum information and thus the estimation of ensembles of multi-qubit states have received particular attention.

A full quantum state estimation for a system composed of n qubits demands the determination of $2^{2n} - 1$ real-valued parameters. The accuracy in estimating each of these parameters decreases rapidly with the system size n to an extent that it becomes difficult to assess the operation of the system even for a moderate number of qubits [1]. In practice, the estimation ensemble is limited, hence it is critical to maximize the amount of information extracted out of each copy.

Substantial work has been targeted towards understanding optimal methods for estimating quantum states under different constraints [2, 3, 4, 5, 6, 7, 8, 9], partly also motivated by understanding the information leakage to an eavesdropper in a quantum communication scenario [10]. The notion of ‘optimal’, however, is ambiguous and depends on the constraints imposed. It might refer to the complexity of the measurement scheme (number of outcomes, projection measurements vs POVMs), to the number of copies consumed for a particular confidence level, or to both. Most theoretical treatments compare the most general collective measurements to projective measurements on individual qubits, and are only valid in the asymptotic limit of infinite copies. This leaves the experimentally more relevant case of small number of copies where POVMs can be performed on each qubit as a largely unexplored regime.

Experimentally, only finite POVMs are feasible, and it is often highly desirable to have a minimal number of outcomes (smallest possible number sufficient to characterize the target Hilbert space). Additionally, it is often not possible to change the measurement from copy to copy, so exactly the same POVM is performed on all members of the ensemble. Thus, we want a minimal POVM that is optimal given its fixed number of measurement outcomes. Such a minimal and optimal POVM was described by Řeháček *et al.* for estimating qubit states [8].

In this paper, we investigate how the accuracy of direct state estimation for the minimal and optimal POVM described in

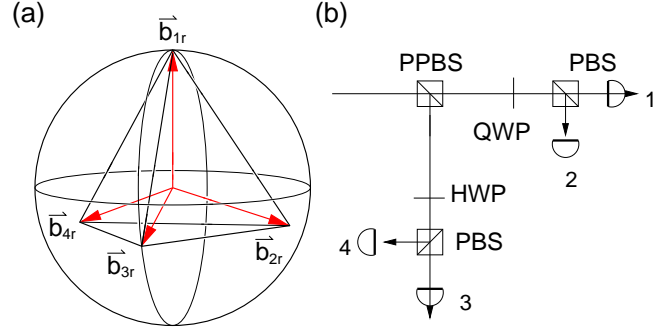


FIG. 1: Optimal polarimetry. (a) To extract the Stokes parameters from an ensemble of photons, polarization measurements are carried out corresponding to four measurement states \vec{b}_j , which form a tetrahedron on the Poincaré sphere. (b) Experimentally, four photon counting detectors are used, each associated with the outcome of one of the measurements. The light is divided between the four detectors by a partially polarizing beam splitter (PPBS), half (HWP)- and quarter (QWP) wave plates, and polarizing beam splitters (PBS) [11].

[8] changes as we increase the number N of copies for one and two-qubit states. Such a scenario is regularly encountered in applications where copies are precious resources, yet the state must be estimated with some accuracy. In this case, it is desirable to know the number of copies that must be sacrificed to reach a certain level of accuracy.

We implement qubit states using the polarization degree of freedom in single photons, and the POVM using polarimeters as in [11, 12], and briefly review the experimental system in Section II. In Section III, we describe a statistical model that gives the accuracy of state estimation for a test state, given any number N of copies. We experimentally verify the statistical prediction by observing maximally polarized one-photon states in Section IV. In Section V, we present observations for similar behavior with two-photon states, as two or more such polarimeters can be used to reconstruct entangled multi-qubit states [12]. Our results suggest a scaling law governing the accurate reconstruction of all multi-qubit states with this method. We conclude in Section VI.

II. MINIMAL AND OPTIMAL POLARIMETRY BY PHOTON COUNTING

There is a close connection between the polarization state of light in classical optics, and the state of a qubit implemented via the polarization states of photons [13]. The macroscopic polarization of light can be thought of as formed by an ensemble of equally prepared photons, where a photon represents the minimum amount of detectable light. A Stokes vector, $\vec{S} = (1, S_1, S_2, S_3)$, describes the macroscopic polarization of light [14], and also is a representation of the density matrix of the qubit state. In classical optics the process of determining the polarization state is called polarimetry [15], and its techniques are often similar to those used in quantum tomography [11].

The main principle of minimal polarimetry is to use the minimum number of measurements to characterize the three free parameters of the Stokes vector, S_1, S_2 and S_3 . Thus, only four measurements are required for complete polarimetry (the fourth one necessary only to obtain a normalization to the total intensity or number of photons present), each corresponding to a particular Stokes vector. If these four Stokes vectors form a tetrahedron in the Poincaré sphere (Figure 1a), then the set of four measurements is minimal and optimal [5], where optimal means that the measurements extract the maximum amount of information from each photon that is detected.

The optimal polarimeter works by partitioning the test ensemble between four detectors (see Figure 1b). A four-by-four instrument matrix B , where each row is one of the measurement vectors, completely characterizes the polarimeter [15]. Detector event distribution and input polarization are connected by the linear form

$$\vec{I} = B \cdot \vec{S}, \quad (1)$$

where $\vec{I} = (I_1, I_2, I_3, I_4)$, with the relative number I_n of events registered by detector n . Knowing B , the observed distribution among the detectors is used to perform linear reconstruction of the input Stokes vector.

We implemented the polarimeter with avalanche photodiodes which are sensitive to single photons. In this way, we can keep track of the photon distribution between the detectors as each copy of the qubit state is detected.

III. A STATISTICAL MODEL FOR PREDICTING THE ACCURACY OF STATE ESTIMATION

A qubit state can only be determined with an infinite number of copies. Experimentally, a good estimate of the state is achieved by collecting an asymptotically large number of copies. We refer to this estimate as the asymptote state. This estimate presents the best possible guess we can make about the unknown input state.

The finite number of copies N limits the accuracy in the estimated state. Although we use the event distribution to directly obtain an estimated state through (1), every distribution

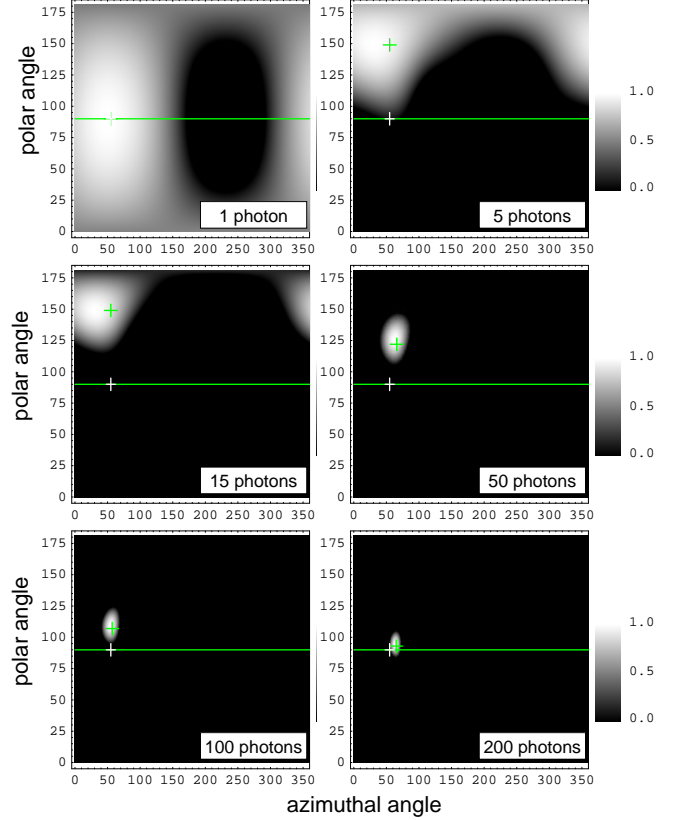


FIG. 2: Projections of the reconstructed probability distribution on the Poincaré sphere surface for a series of measurement outcomes form a polarization state ensemble. The horizontal line in the center of each panel represents linearly polarized states. Light crosses mark the prepared polarization state, and a darker cross marks the estimated (assumed pure) state.

has a deviation from the asymptotic value, and cannot be attributed with absolute certainty to a single Stokes vector \vec{S} . At best, we can assert that the estimated state is compatible with the asymptote state, given the estimated counting uncertainty due to Poisson distribution (assuming independent sequential detection events).

To determine the accuracy of state reconstruction, we consider only specific input states. Since we find experimentally that the accuracy of the worst and best reconstruction cases are quite close, with the counting errors making the two cases indistinguishable, we avoid an analysis of averaging over all possible input states.

The accuracy of an estimated state obtained with a finite ensemble must eventually converge to the accuracy of an asymptotic estimate. Being able to determine the accuracy is useful as it provides a confidence level to an estimated state, based on the number N of detected copies. As a quantitative measure of estimation accuracy, we use the average of the trace distance $D = \frac{1}{2} \text{tr} |\rho_a - \rho_e|$ of an estimated state ρ_e from the asymptote state ρ_a [16], where the averaging occurs over a few consecutive experiments with a fixed number of detection events each.

To model the expected average trace distance for our experimental setting, we must find all the possible ways to distribute N detection events between four detectors. For each partition pattern $k = (n_1, n_2, n_3, n_4)$, there are a total of $c_k = N!/(n_1!n_2!n_3!n_4!)$ compatible detector sequences. Each sequence in a pattern k results in the same reconstructed state \vec{S}_k and consequently same trace distance D_k from the asymptote state. The probability of each sequence is given by

$$p_k = p_1^{n_1} \cdot p_2^{n_2} \cdot p_3^{n_3} \cdot p_4^{n_4}, \quad (2)$$

where p_j is the probability that an input photon will arrive at detector j . Thus, the weighted average \tilde{D} of the trace distance

$$\tilde{D} = \sum_k c_k \cdot p_k \cdot D_k \quad (3)$$

represents the accuracy of the estimated state given N copies.

IV. RECONSTRUCTION OF ONE-QUBIT STATES

A. Maximally polarized or pure one-qubit states

In a first experiment, we consider linear state reconstruction for an increasing number of detected events, converging to an asymptote state. At each stage, we also perform a likelihood estimation to find a region of states that are compatible with the observed photon distribution. The size of this region of states can be interpreted qualitatively as the uncertainty in our estimate.

We used spontaneous parametric down conversion (SPDC) in a non-collinear setup similar to [17] to generate heralded single photons [18] with a controllable polarization state. This allows us to select a well defined ensemble of carriers of the qubit state, virtually unaffected by accidental counts and background noise. Photons were then prepared in a maximally polarized state using a polarization filter and a set of wave plates [19]. We chose to test the polarization state corresponding to the measurement vector $\vec{b}_{1r} = (1, \sqrt{\frac{1}{3}}, \sqrt{\frac{2}{3}}, 0)$ of the polarimeter, because states aligned with the tetrahedral measurement directions are estimated with the poorest accuracy. Vectors anti-aligned with the tetrahedron directions have the best estimation accuracy due to their restricted photon distribution pattern [8]. The worst case scenario should give us a lower bound of the POVM performance; other states should have at least the same accuracy.

We detected 200 copies prepared in this state. For each additional copy, we obtained a state by linear reconstruction under the constraint of the nearest physical state. Concurrently, the likelihood region is determined for a given number of detection events. A projection of both the estimated state and the likelihood region on the surface of the Poincaré sphere is shown in Figure 2 for a selected number of cumulative detection events. Initially, for a small number of available detector outcomes, the estimated state fluctuates strongly, and the likelihood region is large. As the accumulated number of copies increases, the estimated state begins to approach the

asymptote state, and the likelihood region shrinks, indicating a reduction of uncertainty in the estimated state.

B. Accuracy as a function of the detected number of copies

The results of the previous section revealed qualitatively the convergence of an estimated state to the asymptote state. To study the convergence quantitatively, we determine the average trace distance for a given number of detected copies.

We selected three test states: the tetrahedron state \vec{b}_{1r} , its conjugate $-\vec{b}_{1r}$, and the completely unpolarized state $\vec{S} = (1, 0, 0, 0)$. The latter is obtained by collecting light from one arm of the SPDC source without any polarization filters, and is a test for our model for mixed states. The two maximally polarized states represent the worst and best cases in estimating pure states.

For each test state, a very large number of heralded copies (several 10^5) was detected to obtain an approximation to the asymptote state. Next, for each incrementally obtained copy, an estimated state and the corresponding trace distance to the asymptote state was determined. No restrictions were used in obtaining these estimates. Such finite sized collections were repeated 40 times, from which we obtained the average trace distance for each additional copy that was measured.

Selected subsets for the different test states are shown in Figure 3. The average trace distance predicted via (3) is shown by the solid line. The accuracy of the experimental POVM is consistent with the statistical model for both polarized and unpolarized light, and most of the increase in accuracy occurs within the first 100 detected copies. Such a graph can be useful for predicting the average accuracy of any estimation from a finite ensemble of copies. Both analytical and experimental results are consistent with simulations that have been previously reported [8, 9]. We note that our statistical model is also compatible with the prediction in [8] concerning different accuracy levels for different states. However, from any single experimental run, the counting uncertainty make the accuracy levels compatible. Thus, the accuracy of state estimation by the optimal POVM is in practice the very similar for all states.

The results from the statistical model may be analyzed further to reveal how the average trace distance \tilde{D} diminishes with the number of available copies N . A logarithmic representation of the analytical results (see Fig. 4) suggests a power law of the form

$$\tilde{D} = \frac{a}{N^c}. \quad (4)$$

The values for parameters a and c extracted from a least-squares fit to the analytical results for some test states are presented in Table I. The exponent c is close to $1/2$, as perhaps expected from a simple counting statistics argument. Parameter a in (4) seems to represent the difficulty in estimating a particular state; pure polarization states all lead to slightly lower values than the completely unpolarized state.

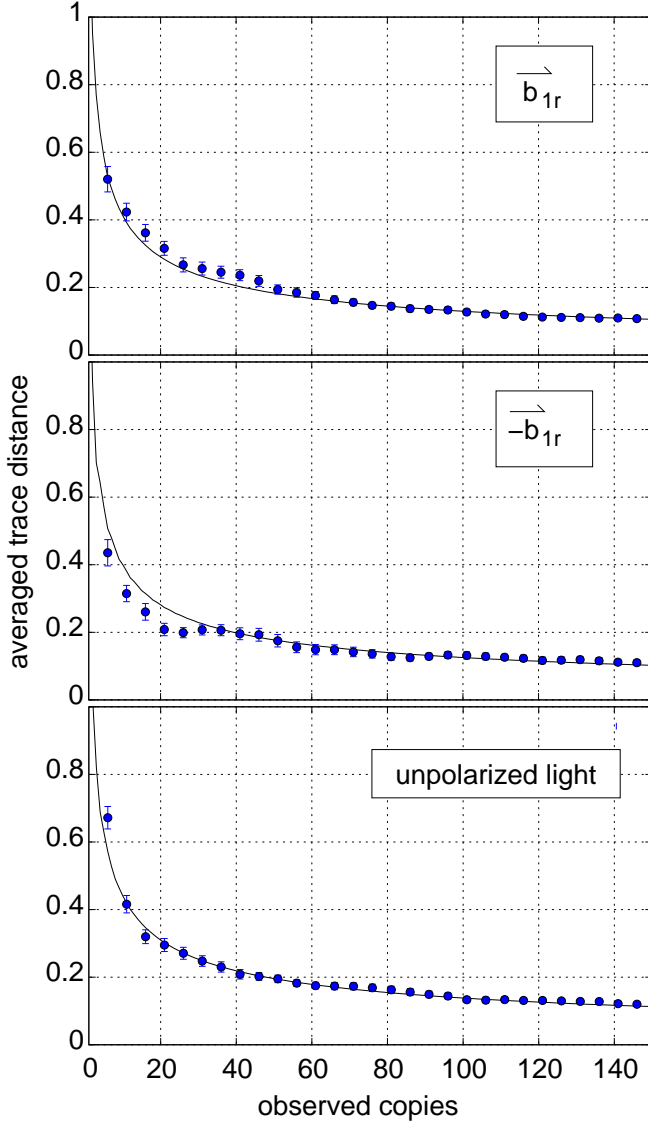


FIG. 3: Trace distance between estimated and asymptotic states for three different input states. The experimental values represent an average over 40 runs with 150 detection events each. Solid curves represent the statistical model (3).

V. ACCURACY IN RECONSTRUCTING A TWO-QUBIT STATE

State estimation of multi-qubit states using only separable measurements and classical communication is readily implemented by a generalization of the previous scheme. For a two photon state, a pair of polarimeters are used simultane-

	unpolarized	horizontal	\vec{b}_{1r}	$-\vec{b}_{1r}$
a	1.417	1.312	1.323	1.288
c	0.506	0.505	0.505	0.506

TABLE I: Power law fit parameters for different test states

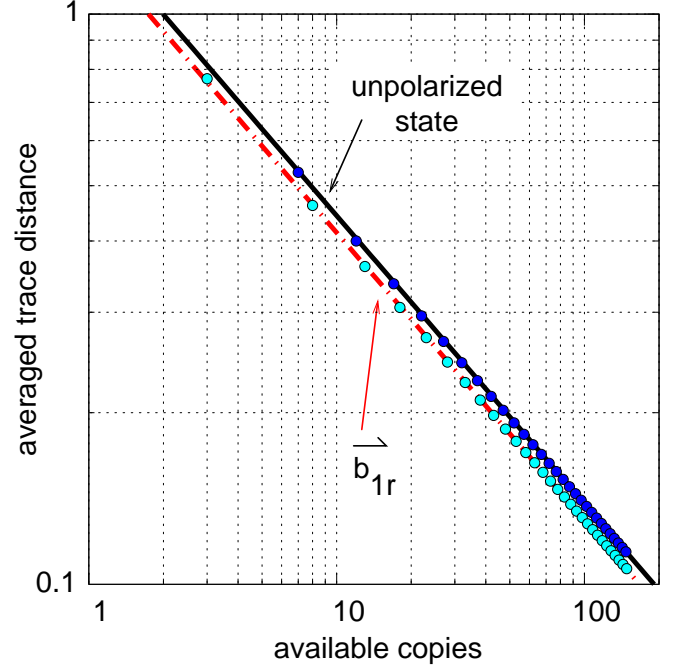


FIG. 4: Average trace distance for two input states using only the results of the statistical model (points). Lines are fits to the power law model (4).

ously to determine the generalized Stokes vector; each photon in the joint state is analyzed by a separate device [12]. By looking at the distribution of coincidence counts between the polarimeters, the two-qubit state, adequately described by a two-photon Stokes vector [11, 20], can be reconstructed.

The photon pairs were prepared in the Bell state $|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|HV\rangle + |VH\rangle)$ with the same SPDC source as in the one-qubit experiments. Several 10^5 copies of the two-qubit state were detected to obtain an estimate of the asymptote state. Then, five sets, each with 5000 detected pairs, were analyzed. Within each set, an estimated state for each incremental pair was obtained, together with its trace distance to the asymptote state. Trace distance values for every cumulative event number were then averaged over those five sets.

To compare the resulting trace distances with the one-photon tests, we introduce a normalized trace distances $\tilde{D}_n = \tilde{D}/(2^{2n} - 1)$ as the trace distance of the (multi)-partite system divided by the number of free parameters n . This normalization attempts to capture the exponential increase in the number of parameters to be determined as the dimensionality increases.

The normalized results are compared in Figure 5. We find that for both one- and two-photon systems, the average trace distance is within 0.01% after 5000 detection events, and appears to follow the same dependence on the number of copies N . This suggests that the separable POVM reconstructs multi-qubit states with the same accuracy per copy and dimension of the multi-qubit state, providing possibly a simple scaling law to determine the number of copies necessary to estimate a multi-qubit system to any given accuracy. It also provides an

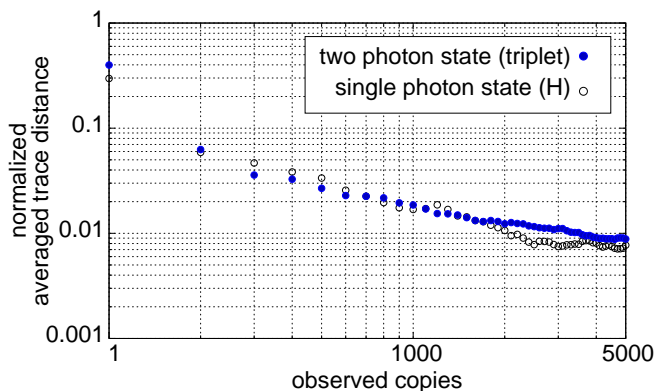


FIG. 5: Averaged trace distance (over five experimental runs) for a finite number of detection events for the two-qubit Bell state $|\Psi^+\rangle$ and a horizontally polarized one-qubit state, normalized to the number of free parameters (3 for a one-photon state and 15 for the two-photon state).

experimentally relevant benchmark to evaluate different measurement strategies.

VI. SUMMARY

We have presented an experimental observation of the average accuracy of a minimal and optimal POVM, taking into

account the uncertainty due to a finite number of available copies of a state. The average trace distance was identified as a measure of accuracy, and the reduction in trace distance with the number of detected copies was studied. The results were compatible with a simple model for any number of detected copies using multinomial statistics.

The estimation uncertainty for different input states was compatible with Poissonian counting statistics for realistic experimental situations. With this view, the POVM method is able to estimate all single-qubit states equally well.

Furthermore, the normalized accuracy in estimating a two-qubit state appears to follow the same scaling law as in the one-qubit case, suggesting a scaling law for all multi-partite systems.

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